Subject: Leaving Certificate Maths

Teacher: Mr Murphy

Lesson 22: Complex Numbers I

22.1 Learning Intentions

After this week's lesson you will be able to;

- Explain the need for complex numbers
- Outline the relationship to the other types of numbers
- Explain what an Argand diagram is
- Calculate the modulus of a complex number
- Transformations of a complex number
- Adding/Subtracting
- Multiplying
- Division

22.2 Specification

– investigate the operations of addition, multiplication, subtraction and division with complex numbers **C** in rectangular form a+ib
– illustrate complex numbers on an Argand diagram
– interpret the modulus as distance from the origin on an Argand diagram and calculate the complex conjugate

22.3 Chief Examiner's Report

A	4	13·3	40	9	Complex numbers in fraction form

22.4 Why do we need these numbers?

Let's look at the following equation:

$$x^2 = 25$$
$$\boldsymbol{x = \sqrt{25}}$$
$$x = \pm 5$$

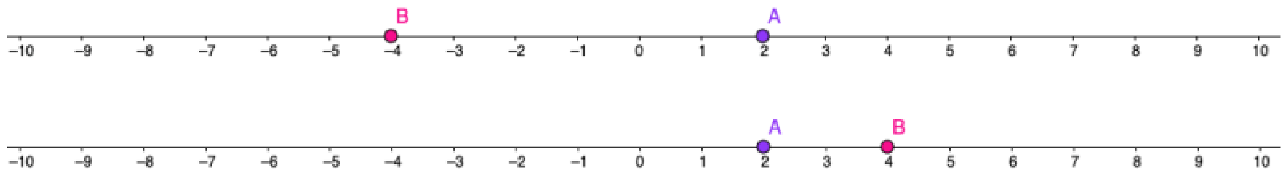
What about now?

$$x^2 = -25$$
$$\boldsymbol{x = \sqrt{-25}}$$

So, there is a need to reimagine our number systems. To do this let's go back to the number line.

22.5 Figuring Out Our New Numbers

Number Line:



We can now view the concept of multiplication as a transformation, in particular a rotation and a change in magnitude.

If we take a particular number, n, and multiply it by -4 we can say it quadruples in magnitude and rotates 180°. If we continue with this multiplication, we can get how many rotations we have and a scale factor.

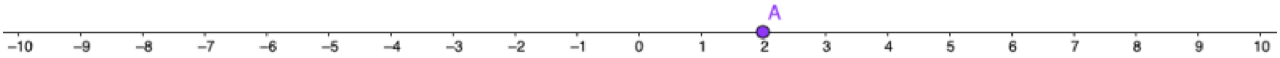
$$n \times (-4) \qquad 180° \qquad (-4)$$

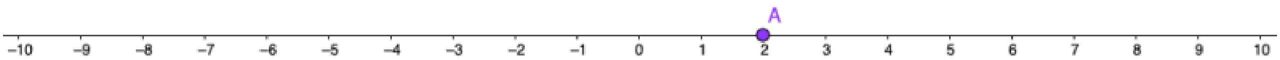
$$n \times (-4)^2 \qquad 180° + 180° \qquad (-4)^2$$

$$n \times (-4)^3 \qquad 180° + 180° + 180° \qquad (-4)^3$$

Write the pattern down below:

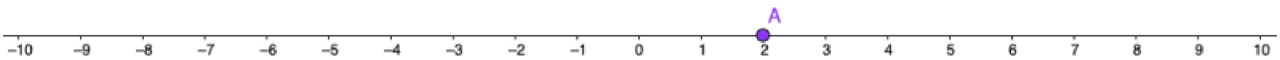
Let’s look at a different number now:





So, what can we now say about multiplying by $(-1)^n$

What if $n \in \mathbb{Q}$?

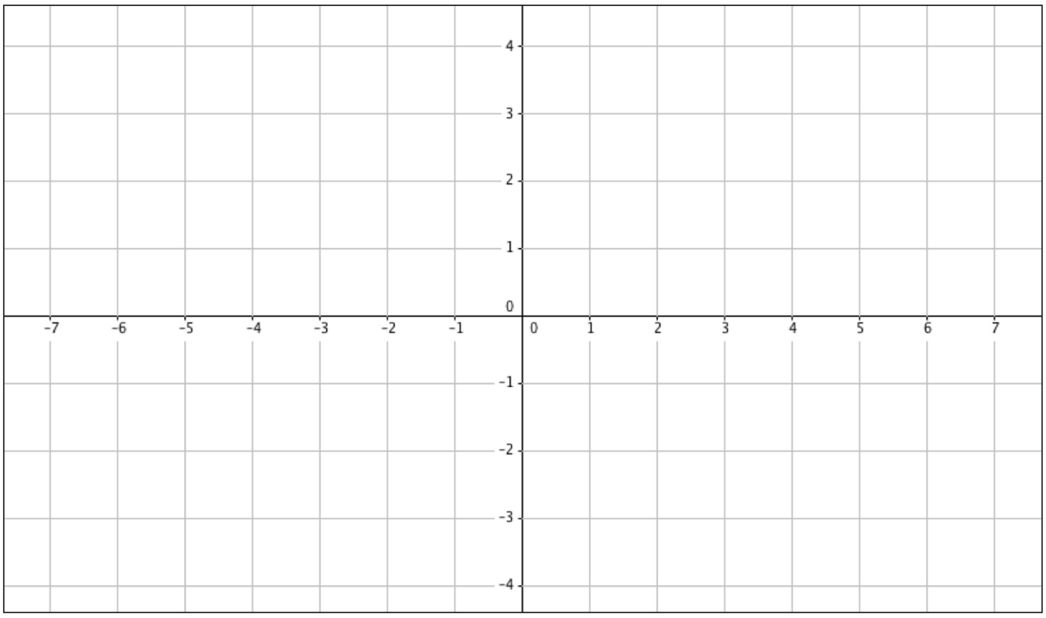


22.6 Argand Diagram

As we have seen in 22.5 there is this new region rising out of the real plane, known as the imaginary axis.

Similar to Cartesian coordinates, a point on the complex plane has an x and y part. Therefore, on an Argand diagram the number has a real part, Re(z) and an imaginary part, Im(z).

The real part is the constant segment of the number and the imaginary part is the term containing the i. These two together make up our new number system, complex numbers.



22.7 Adding/Subtracting Complex Numbers

This will often be seen as a similar operation to what we do in algebra, gather like terms. With this in mind we add the real parts of the number and then add the imaginary parts to get our result. Let us not forgot the visual representation we now have for complex numbers, the Argand Diagram.

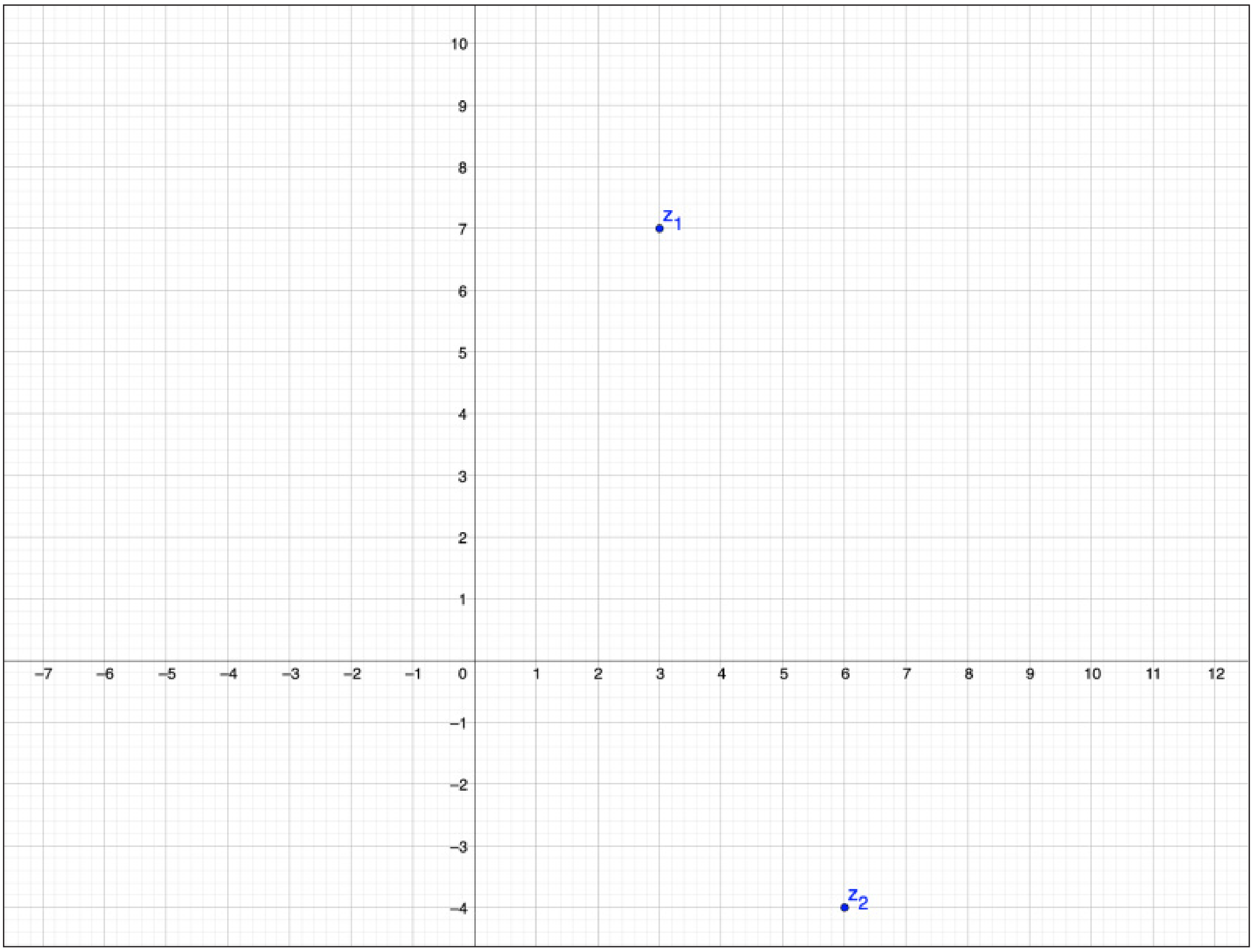
Question:

$If\ z_1 = 3 + 7i\ and\ z_2 = 6 - 4i\ find\ the\ following:$

$z_1 + z_2 =$

$z_1 - z_2 =$

Let's have a look at this visually:



22.8 Multiplication of Complex Numbers

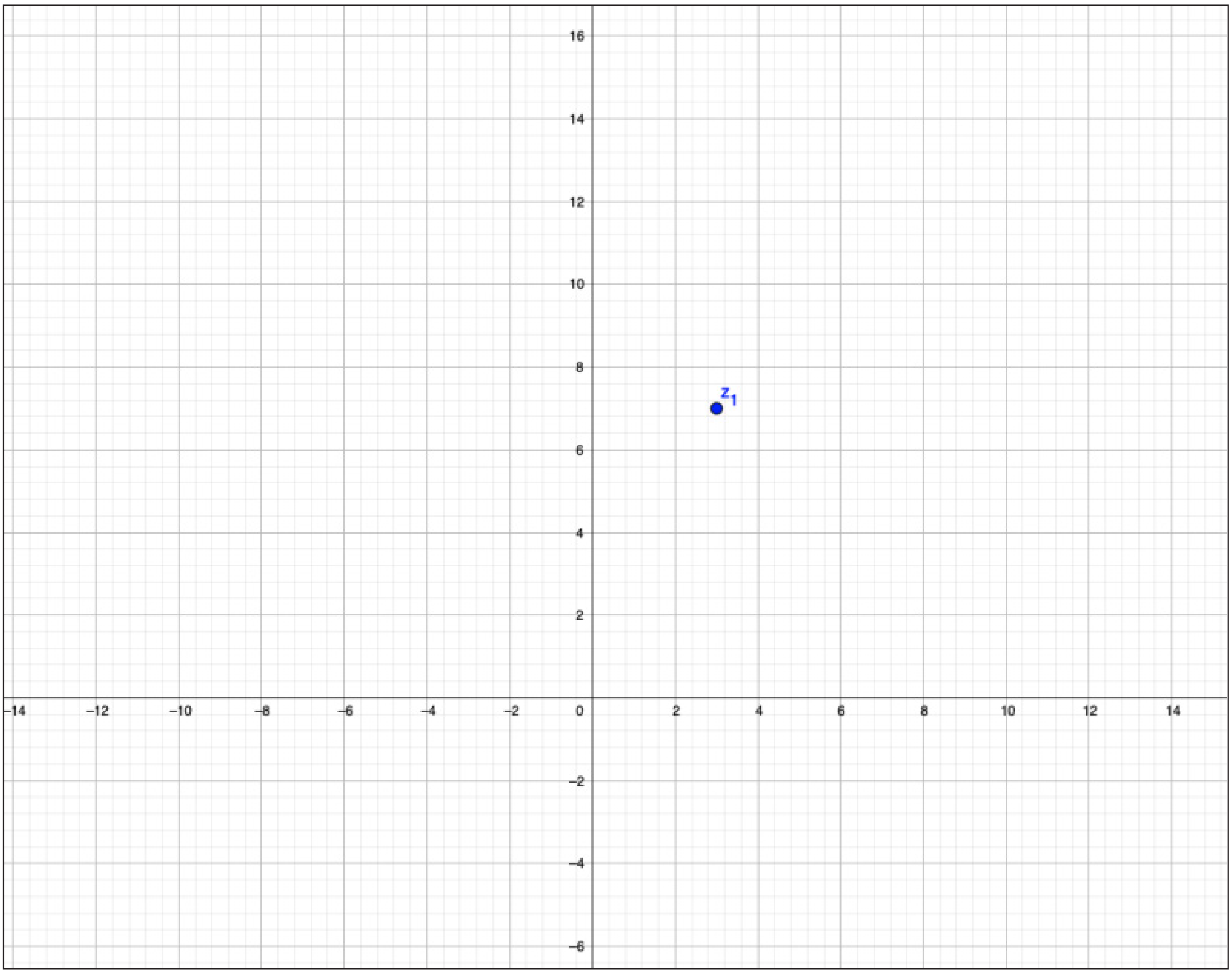
To fully understand this operation, we must look at two ideas separately. What happens if we multiply by only a real number and then if we multiply by an imaginary number.

$z_1 = 3 + 7i$

What is $2z_1$?

What is iz_1?

Argand Diagram:



Based on the above we can see that multiplying by a real number affects the scale of the complex number only.

Multiplying by an imaginary number (i) causes a 90° rotation anti-clockwise.

What about multiplying by a negative i?

What is $-iz_1$?

What is $-2iz_1$?

$\boldsymbol{If\ \ z_2 = 6 - 4i\ find\ the\ following}$:

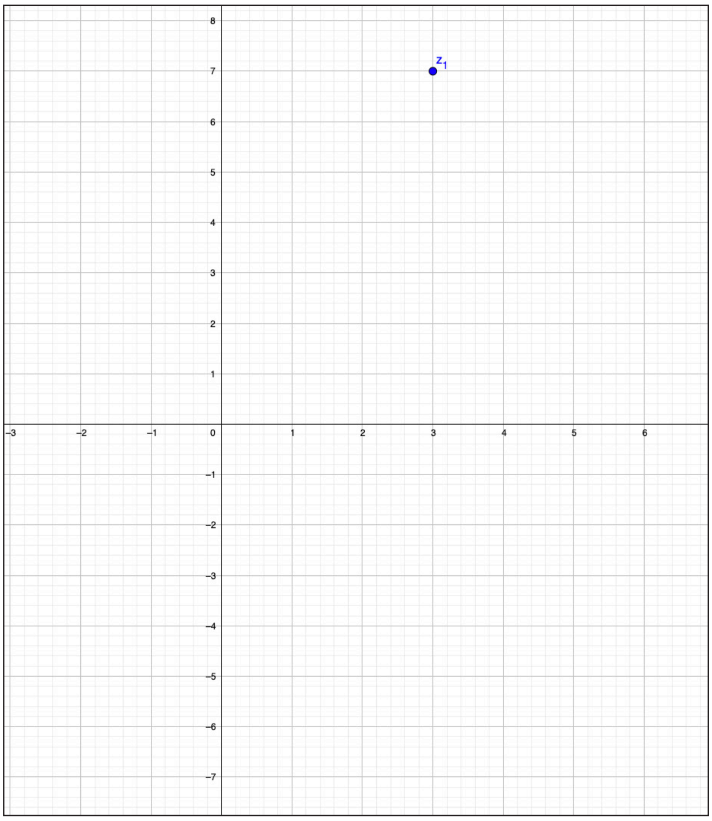
i) z_1z_2

22.9 Division of Complex Numbers

To first understand division of these numbers we must introduce something called the complex conjugate of a complex number. In terms of notation we have:

$$If\ z = a + bi,\ then\ \bar{z} = a - bi$$

In terms of a visual representation we have:



The complex conjugate is a reflection of the original complex number about the x-axis.

If we multiply a complex number by its conjugate, we have:

$$If\ z = a + bi\ and\ \bar{z} = a - bi$$
$$z\bar{z} = (a + bi)(a - bi)$$
$$z\bar{z} = a^2 - abi + abi - (bi)^2$$
$$z\bar{z} = a^2 - (b^2 i^2)$$
$$z\bar{z} = a^2 - \left(b^2(-1)\right)$$
$$z\bar{z} = a^2 - (-b^2)$$
$$z\bar{z} = a^2 + b^2$$

Which is actually the same as the modulus of z squared or $(|z|)^2$

So, this gives us a real number as a result. This is helpful in division to remove the imaginary number as a denominator.

$$If\ z_1 = 3 + 7i\ and\ z_2 = 6 - 4i\ find\ the\ following:$$

$$\frac{z_1}{z_2} =$$

22.10 Recap of the Learning Intentions

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22.11 Homework Task

(a) The complex numbers z_1, z_2 and z_3 are such that $\frac{}{z_1} = \frac{}{z_2} + \frac{}{z_3}$, $z_2 = 2 + 3i$ and $z_3 = 3 - 2i$, where $i^2 = -1$. Write z_1 in the form $a + bi$, where $a, b \in \mathbb{Z}$.

22.12 Solutions to 21.7

Prove by induction that $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$

Step 1: Check $n = 1$, so, is the sum of the first 1 natural number equal to 1?

$$\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1$$

Statement is true for n = 1.

Step 2: Assume true for $n = k$, *where* $k \geq 1$.

$\sum_{r=1}^{n} r = 1 + 2 + 3 \ldots + n.$ *Subbing this in we get:*

$$1 + 2 + 3 \ldots + k = \frac{k(k+1)}{2}$$

Step 3: Show true for $n = k + 1$, *where* $k \geq 1$.

$$1 + 2 + 3 \ldots + k + k + 1 = \frac{k(k+1)}{2} + k + 1$$
$$1 + 2 + 3 \ldots + k + k + 1 = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$
$$1 + 2 + 3 \ldots + k + k + 1 = \frac{k(k+1) + 2k + 2}{2}$$
$$1 + 2 + 3 \ldots + k + k + 1 = \frac{k^2 + k + 2k + 2}{2}$$

$1 + 2 + 3 \ldots + k + k + 1 = \frac{k^2+3k+2}{2}$ ... subbing in from step two on the LHS

$$\frac{k(k+1)}{2} + k + 1 = \frac{(k+2)(k+1)}{2}$$

$$\frac{(k+2)(k+1)}{2} = \frac{(k+2)(k+1)}{2}$$

Step 4: As this was true for $n = k$, *this is also true for* $n = k + 1$.